On the viscoelastic characterization of the Jeffreys–Lomnitz law of creep

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Abstract

In 1958 Jeffreys proposed a power law of creep, generalizing the logarithmic law earlier introduced by Lomnitz, to broaden the geophysical applications to fluid-like materials including igneous rocks. This generalized law, however, can be applied also to solid-like viscoelastic materials. We revisit the Jeffreys-Lomnitz law of creep by allowing its power law exponent α, usually limited to the range $0 \leq \alpha \leq 1$ to all negative values. This is consistent with the linear theory of viscoelasticity because the creep function still remains a Bernstein function, that is positive with a completely monotone derivative, with a related spectrum of retardation times. The complete range $\alpha \leq 1$ yields a continuous transition from a Hooke elastic solid with no creep ($\alpha \to -\infty$) to a Maxwell fluid with linear creep ($\alpha = 1$) passing through the Lomnitz viscoelastic body with logarithmic creep ($\alpha = 0$), which separates solid-like from fluid-like behaviors. Furthermore, we numerically compute the relaxation modulus and provide the analytical expression of the spectrum of retardation times corresponding to the Jeffreys-Lomnitz creep law extended to all $\alpha \leq 1$.

Key Words and Phrases: Creep, Relaxation, Linear Viscoelasticity, Jeffreys-Lomnitz law, Completely monotone functions, Bernstein functions, Laplace Transform.

1 Introduction

According to the linear uni-axial theory of viscoelasticity, a material body can be viewed as a linear system where either stress $\sigma(t)$ or strain $\epsilon(t)$ have the role of excitation function (input) and response function (output). From both the experimental and theoretical viewpoints, a central role is played by the *creep* and *relaxation tests*, where the inputs are a step-wise stress and strain, respectively. Introducing the Heaviside step function $\Theta(t)$, in a creep test $\sigma(t) = \sigma_0\Theta(t)$, conversely in a relaxation test $\epsilon(t) = \epsilon_0\Theta(t)$, with constants σ_0 and ϵ_0. The corresponding outputs are characterized by the so-called *time-dependent material functions*, the *creep compliance $J(t) = \epsilon(t)/\sigma_0$* and the *relaxation modulus $G(t) = \sigma(t)/\epsilon_0$*, respectively. From experimental evidence, $J(t)$ is non-decreasing and non-negative, while $G(t)$ is non-increasing and non-negative.

In Earth rheology, transient creep is often described by the Jeffreys-Lomnitz law. This law of creep is defined in terms of four parameters as follows

$$J(t) = J_0[1 + q\psi(t)], \quad t \geq 0, \tag{1}$$

where J_0 is the unrelaxed compliance, q is a positive dimensionless material constant, and

$$\psi(t) = \frac{\left(1 + \dfrac{t}{\tau_0}\right)^\alpha - 1}{\alpha}, \quad 0 \leq \alpha \leq 1, \quad t \geq 0. \tag{2}$$

Here, τ_0 is a characteristic time and the exponent α essentially determines the rheological behaviour of the creep law. By its own definition, $\psi(t)$ is a positive increasing function of time that we refer to as the dimensionless *Jeffreys-Lomnitz creep law*.

Creep law (1) was proposed by Sir Harold Jeffreys in 1958 [24] (see also [25] and [26, 27]), to generalize, via the parameter α, the logarithmic law

$$\psi(t) = \log\left(1 + \frac{t}{\tau_0}\right), \quad t \geq 0. \tag{3}$$

introduced by Lomnitz in 1956 [29] to describe flow in igneous rocks. This law was also employed by Lomnitz to account for the damping of the Earth's free nutation (Chandler wobble) and of seismic S-waves, see Lomnitz (1957, 1962) [30, 31].

As a matter of fact, for $\alpha = 0$, the Jeffreys law reduces (in the limit) to the Lomnitz logarithmic law; while for $\alpha = 1$, it reduces to the simple linear law of the Maxwell viscoelastic body. According to Jeffreys, Eq. (2) better fits the data of creep and dissipation in the Earth for seismological purposes. The Jeffreys-Lomnitz law is indeed well recognized in rheology of rocks, in view of its property of interpolating creep data between a logarithmic and a linear law, see *e.g.* Jeffreys [25] and Ranalli [38].

Although in the original papers the exponent α was not explicitly limited to positive values, the applications with $\alpha < 0$ were only consequently introduced in Earth rheology. We note that the late Professor Ellis Strick in his 1984 paper [43] had already extended the Jeffreys-Lomnitz law of creep allowing the parameter α in the range $-1 \leq \alpha \leq +1$, by introducing parameter $s := 1 - \alpha$ ($0 \leq s \leq 2$). Furthermore, he was also interested in the representation of the extended Jeffreys-Lomnitz law of creep in terms of a suitable ladder network of springs and dashpots. In his work, Strick was motivated by some experimental observations suggesting negative values of the exponent α. At the time we started editing the present paper, we were not aware of any reaction of the geophysical community to his results. However, later we noted some published works, see *e.g.* Crough & Burford (1977) [8], Spencer (1981) [41], Wesson (1988) [50], Darby & Smith (1990) [9], where the Jeffreys-Lomnitz law of creep was applied with $\alpha < 0$ to fit experimental geophysical data. In these papers, however, we did not find any attempt to characterize the viscoelastic properties of the Jeffreys-Lomnitz law in the whole range $-\infty < \alpha \leq 1$. In particular, the relaxation modulus and the retardation spectrum associated to this creep law have never been investigated up to now. Of course, in the trivial case $\alpha = 1$, we recover the Maxwell body for which the retardation spectrum vanishes and the relaxation modulus is exponentially decreasing.

The purpose of this paper is to characterize the Jeffreys-Lomnitz law of creep consistently with the linear theory of viscoelasticity. In order to meet this goal, we find it convenient for the readers to recall in Section 2 the essential mathematical notions of linear viscoelasticity necessary to understand our analysis. Then, in Section 3, we consider this law of creep allowing the parameter α in the whole range $-\infty < \alpha \leq 1$. Here, for some selected values of α in the range $-2 \leq \alpha \leq 1$, we show plots of the analytically determined creep function and of the corresponding relaxation function, numerically computed by solving an integral Volterra equation

of the second kind. In Section 4, we analytically derive the spectrum of retardation times corresponding to the extended creep law by using the tools of Laplace transforms, and we exhibit plots for the above selected values of α. In Section 5, we draw our conclusions and final remarks. We have assumed that the reader is familiar with the main properties and tables of Laplace transforms so we do not recall them. However, we devote the Appendix to the Post-Widder formula of Laplace inversion on the real axis, which yields a complementary approach to our derivation of the retardation spectrum.

2 Essentials of linear viscoelasticity

The basic principles of linear viscoelasticity are illustrated in well known treatises [20, 6, 18, 7, 36, 46, 28]. Here we mostly follow the notation of Chapter 2 of the recent book of Mainardi [33], where the approaches by Gross [20], Pipkin [36] and Tschoegl [46] have been mainly adopted and revisited. Without loss of generality, the one-dimensional theory will be employed throughout.

Assuming causal and differentiable histories of stress and strain (i.e., $\sigma(t)$ and $\epsilon(t)$ vanish for all times $t < 0$ and are differentiable for $t > 0$), the time-dependent material functions formerly introduced, $J(t)$ and $G(t)$, can be used to express the general stress-strain relationship as

$$\epsilon(t) = \sigma(0^+) \, J(t) + \int_0^t J(t - \tau) \, \dot{\sigma}(\tau) \, d\tau = \sigma(0^+) \, J(t) + J(t) * \dot{\sigma}(t), \quad (4)$$

or, equivalently

$$\sigma(t) = \epsilon(0^+) \, G(t) + \int_0^t G(t - \tau) \, \dot{\epsilon}(\tau) \, d\tau = \epsilon(0^+) \, G(t) + G(t) * \dot{\epsilon}(t), \quad (5)$$

where dots and $*$ denote time-differentiation and time-convolution, respectively.

The limiting values of the time-dependent material functions for $t \to 0^+$ and $t \to +\infty$ describe the instantaneous and equilibrium behaviour of the viscoelastic body, respectively. The instantaneous J_0 and the equilibrium compliance J_∞ are defined by $J_0 \equiv J(0^+)$ and $J_\infty \equiv J(+\infty)$, while the instantaneous and equilibrium modulus are $G_0 \equiv G(0^+)$ and $G_\infty \equiv G(+\infty)$, respectively.

Taking the Laplace transform of Eqs. (4) and (5), the following *reciprocity relation* is obtained

$$s\,\widetilde{J}(s) = \frac{1}{s\,\widetilde{G}(s)} \quad \text{or} \quad \widetilde{J}(s)\,\widetilde{G}(s) = \frac{1}{s^2}, \tag{6}$$

where s is the Laplace complex variable and the tilde denotes transformed quantities[1]. Applying the limiting theorems of the Laplace transform in the first of Eqs. (6) provides

$$J_0 = \frac{1}{G_0} \quad \text{and} \quad J_\infty = \frac{1}{G_\infty}, \tag{7}$$

which allow us for a classification of the viscoelastic bodies into four types, according to the values of moduli and compliances (Table 1).

Table 1: Classification of the four types of viscoelastic bodies according to values of J_0, J_∞, G_0 and G_∞ (conventionally, $1/0 = \infty$).

Type	J_0	J_∞	G_0	G_∞
I	> 0	$< \infty$	$< \infty$	> 0
II	> 0	$= \infty$	$< \infty$	$= 0$
III	$= 0$	$< \infty$	$= \infty$	> 0
IV	$= 0$	$= \infty$	$= \infty$	$= 0$

Type I bodies exhibit both instantaneous and equilibrium elasticity, so they behave similarly to an elastic (Hooke) body for sufficiently short and long times. Bodies of type II and IV show a complete stress relaxation (at constant strain) since $G_\infty = 0$ and an infinite strain creep (at constant stress) since $J_\infty = \infty$,, so they do not exhibit equilibrium elasticity. Finally, bodies of type III and IV do not present instantaneous elasticity since $J_0 = 0$ $(G_0 = \infty)$, hence they will not be considered in this study. By the Laplace inversion

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[1]For the most relevant properties of the Laplace transform, we refer, e.g,. to the Appendix A3 (Eq. 32) of the well-known treatise by Tschoegl, see [46] pp. 560–570. In the following we use the notation $f(t) \div \widetilde{f}(s)$ to denote the juxtaposition of an original function $f(t)$ with its Laplace transform $\widetilde{f}(s)$.

of the second of Eqs. (6), we obtain the interrelation between the material functions in the time domain

$$J(t) * G(t) := \int_0^t J(t - t')\, G(t')\, dt' = t, \quad t \geq 0, \tag{8}$$

which will be useful in the following to obtain, by numerical methods, $G(t)$ from the knowledge of $J(t)$ in the case of the (generalized) Jeffreys-Lomnitz law of creep. In fact, if $J_0 > 0$ (types I and II), Eq. (8) can be re-written as a Volterra integral equation of the second kind

$$G(t) = \frac{1}{J_0} - \frac{1}{J_0} \int_0^t \dot{J}(t - t')\, G(t')\, dt', \tag{9}$$

with $G(t)$ unknown. Of course, a similar equation can be established for $J(t)$, provided that $G(t)$ is known and $G_0 < \infty$ (types I and II).

Another relevant consequence of Eq. (8) is

$$J(t)\, G(t) \leq 1, \quad t \geq 0, \tag{10}$$

where the equality holds in the limiting cases $t \to 0^+$ and $t \to +\infty$. The results in Eqs. (9), (10) are discussed in [36] and recently revisited in [33].

We remark the restrictive conditions related to the monotonicity of the time-dependent material functions. According to Gross [20], their general expressions consistent with experimental observations are as follows:

$$\begin{cases} J(t) = J_0 + \int_0^\infty R_\epsilon(\tau) \left(1 - e^{-t/\tau}\right) d\tau + J_+ \, t, \\ G(t) = G_\infty + \int_0^\infty R_\sigma(\tau)\, e^{-t/\tau}\, d\tau + G_-\, \delta(t), \end{cases} \tag{11}$$

where the non-negative functions $R_\epsilon(\tau)$ and $R_\sigma(\tau)$ denote the *retardation spectrum* and the *relaxation spectrum*, respectively. For the classical mechanical models, obtained by series and parallel combinations of a finite number of springs and dashpots [33], the spectra are discrete.

It is convenient to consider separately, for $t \geq 0$, those terms of (11) deriving from a continuous spectrum

$$\begin{cases} J_\tau(t) := \int_0^\infty R_\epsilon(\tau) \left(1 - e^{-t/\tau}\right) d\tau, \\ G_\tau(t) := \int_0^\infty R_\sigma(\tau)\, e^{-t/\tau}\, d\tau, \end{cases} \tag{12}$$

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where $J_\tau(t)$ (the *creep function with spectrum*) is a non-decreasing, non-negative function with limiting values $J_\tau(0^+) = 0$ and $J_\tau(+\infty) \leq \infty$, whereas $G_\tau(t)$ (the *relaxation function with spectrum*) is a non-increasing, non-negative function, with $G_\tau(0^+) \leq \infty$ and $G_\tau(+\infty) = 0$. The integrability on $\mathbb{R}^+$ of $R_\epsilon(\tau)$ and $R_\sigma(\tau)$ is ensured if $J_\tau(+\infty) < \infty$.

We also note that, consistently with the spectral representations (12), we have $(-1)^n J_\tau^{(n)}(t) \leq 0$ and $(-1)^n G_\tau^{(n)}(t) \geq 0$, where the superscript denotes the n–th time derivative. From a mathematical standpoint, these conditions are equivalent to require that $J_\tau(t)$ and $G_\tau(t)$ are *Bernstein* and *complete monotone functions*, respectively. For details, we refer to specialized mathematical treatises, Berg and Forst [5], Gripenpeg et al. [19], and Schilling et al. [40]. These properties have been investigated by several authors, including Molinari [34], Del Piero and Deseri [10] and, more recently, by Hanyga [21].

The determination of the *time-spectral functions* from the knowledge of the time-dependent material functions is a problem that can be formally solved through a method, outlined by Gross [20], based on *Laplace transform pairs*. For this purpose we introduce the *frequency-spectral functions* as

$$S_\epsilon(\gamma) := \frac{R_\epsilon(1/\gamma)}{\gamma^2}, \quad S_\sigma(\gamma) := \frac{R_\sigma(1/\gamma)}{\gamma^2}, \tag{13}$$

where $\gamma = 1/\tau$ denotes a retardation or relaxation frequency. Noting that $R_\epsilon(\tau)\,d\tau = S_\epsilon(\gamma)\,d\gamma$ and $R_\sigma(\tau)\,d\tau = S_\sigma(\gamma)\,d\gamma$, time differentiation of (12) yields

$$\begin{cases} + \dot{J}_\tau(t) = \displaystyle\int_0^\infty \frac{R_\epsilon(\tau)}{\tau}\,\mathrm{e}^{-t/\tau}\,d\tau = \int_0^\infty \gamma\,S_\epsilon(\gamma)\,\mathrm{e}^{-t\gamma}\,d\gamma\,, \\[4mm] -\dot{G}_\tau(t) = \displaystyle\int_0^\infty \frac{R_\sigma(\tau)}{\tau}\,\mathrm{e}^{-t/\tau}\,d\tau = \int_0^\infty \gamma\,S_\sigma(\gamma)\,\mathrm{e}^{-t\gamma}\,d\gamma\,, \end{cases} \tag{14}$$

showing that $\gamma\,S_\epsilon(\gamma)$ and $\gamma\,S_\sigma(\gamma)$ can be viewed as the inverse Laplace transforms of $\dot{J}_\tau(t)$ and $-\dot{G}_\tau(t)$, respectively, where t is now considered the Laplace transform variable instead of the usual s. Thus, adopting the connective symbol $\div$ for Laplace transform pairs where in the LHS we put

the original function and in the R.H.S. its Laplace transform, we have

$$\begin{cases} +\gamma\,S_\epsilon(\gamma) = \dfrac{R_\epsilon(1/\gamma)}{\gamma}\;\div\;\dot{J}_\tau(t)\,, \\[4mm] -\gamma\,S_\sigma(\gamma) = \dfrac{R_\sigma(1/\gamma)}{\gamma}\;\div\;\dot{G}_\tau(t)\,. \end{cases} \tag{15}$$

Consequently, when $J_\tau(t)$ and $G_\tau(t)$ are known analytically, the corresponding frequency-spectral functions can be derived by standard methods for the inversion of Laplace transforms; then, from Eq. (13), the time-spectral functions are easily obtained.

3 The extended Jeffreys-Lomnitz laws of creep and relaxation

In this Section we revisit the Jeffreys-Lomnitz law of creep by extending the range of variability of the exponent α to any negative value in Eq. (2). In the expression for the dimensionless Jeffreys-Lomnitz creep law, it is now convenient to separately consider four cases:

$$t \geq 0\,, \quad \psi(t) = \begin{cases} t/\tau_0\,, & \alpha = 1\,, \\[4mm] \dfrac{(1 + t/\tau_0)^\alpha - 1}{\alpha}\,, & 0 < \alpha < 1\,, \\[4mm] \log(1 + t/\tau_0)\,, & \alpha = 0\,, \\[4mm] \dfrac{1 - (1 + t/\tau_0)^{-|\alpha|}}{|\alpha|} & \alpha < 0\,. \end{cases} \tag{16}$$

The behaviour of $\psi(t)$ as a function of the dimensionless time t/τ_0 is illustrated in Figure 1, for some values of α in the range $-2 \leq \alpha \leq 1$, adopting a logarithmic and a linear time axis in the top and in the bottom frames, respectively. From Eq. (16), the different asymptotic behaviour of $\psi(t)$ as $t \to \infty$ in the cases $0 \leq \alpha \leq 1$ and $\alpha < 0$ is apparent. In the former case, $\psi(t)$ diverges whereas in the latter case it increases up to the finite value $1/|\alpha|$.
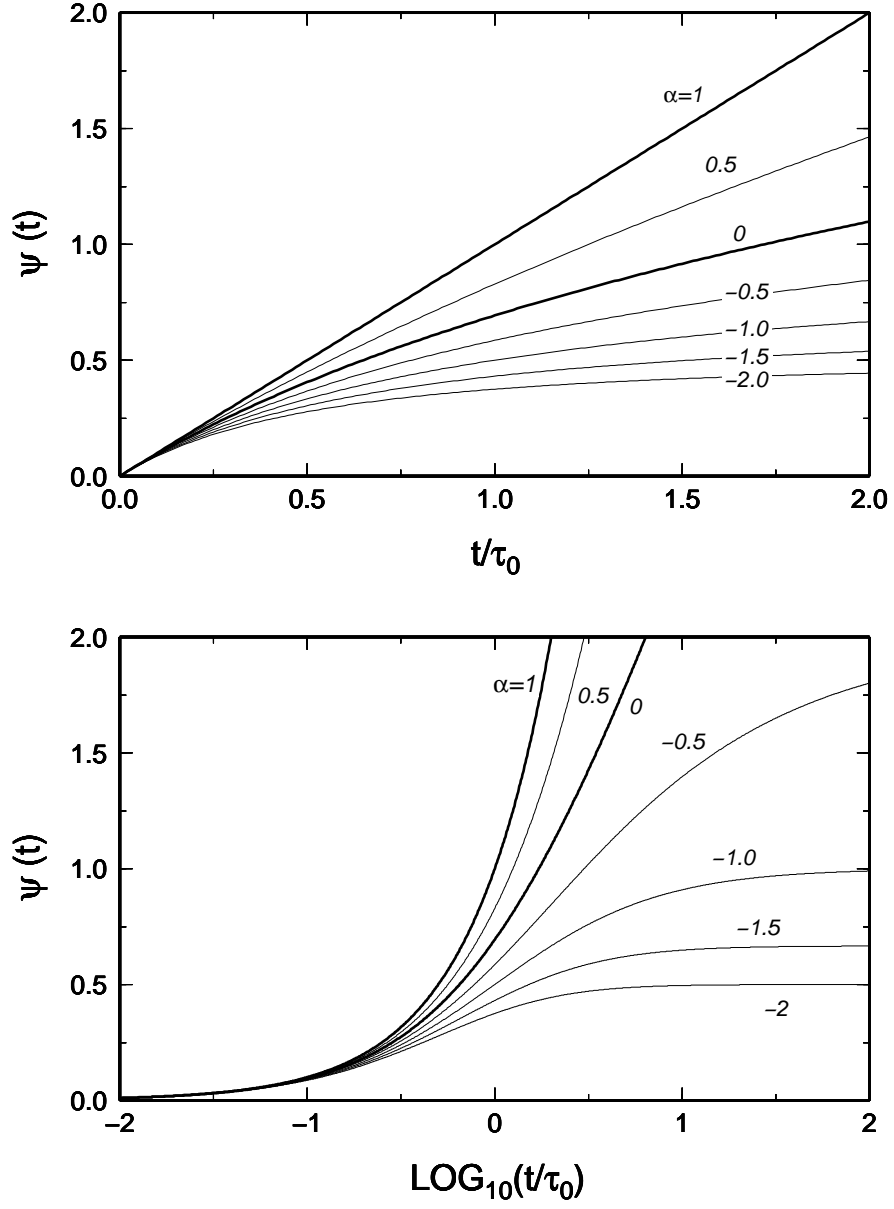
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Figure 1: The creep function $\psi(t)$ for $\alpha = 1, 0.5, 0, -0.5, -1, -1.5, -2$ versus dimensionless time t/τ_0, adopting a linear scale (top) and a logarithmic scale (bottom).

In consideration of the asymptotic behaviour of $\psi(t)$, from the classification of the types of viscoelasticity of Table 1, we recognize that the law provided by Eq. (16), along with (1), describes the behavior of a viscoelastic body that shows instantaneous elasticity. For $\alpha < 0$ and $0 \leq \alpha \leq 1$, it falls in types I and II, respectively.

Since the extended Jeffreys-Lomnitz law accounts for instantaneous elasticity, it is possible to write the *relaxation modulus* as follows

$$G(t) = G_0\phi(t)\,, \quad t \geq 0\,, \tag{17}$$

where $0 < G_0 = 1/J_0 < \infty$ and $\phi(t)$ is a positive decreasing function of time that we refer to as the dimensionless *relaxation function*.

Using Eqs. (1) and (17) in (6), the following interrelation in the Laplace domain is found between the dimensionless creep and relaxation functions:

$$s\,\widetilde{\phi}(s) = \frac{1}{1 + qs\,\widetilde{\psi}(s)}\,, \quad \text{or} \quad \widetilde{\phi}(s) = \frac{1/s}{1 + qs\widetilde{\psi}(s)}\,, \tag{18}$$

where $\widetilde{\phi}(s)$ and $\widetilde{\psi}(s)$ are the Laplace transforms of $\phi(t)$ and $\psi(t)$, respectively. From the limiting theorems of Laplace transforms applied to the first of Eqs. (18), we easily obtain the following:

$$\phi(0^+) = 1\,, \quad \phi(+\infty) = \begin{cases} 0\,, & 0 \leq \alpha \leq 1\,, \\ \dfrac{1}{1 + q/|\alpha|}\,, & \alpha < 0\,. \end{cases} \tag{19}$$

However, the analytical derivation of $\phi(t)$ from inverting the Laplace transforms in Eqs. (18) appears as a difficult (if not prohibitive) task even if $\widetilde{\psi}(s)$ can be expressed in terms of a transcendental function related to incomplete Gamma function, see [43]. In order to obtain the relaxation function, we numerically solve the Volterra integral equation of the second kind (9) with $J(t)$ appropriate for the extended Jeffreys-Lomnitz law. After straightforward algebra, the Volterra equation reads:

$$\phi(t) = 1 - q \int_0^t \left(1 + \frac{t - t'}{\tau_0}\right)^{\alpha - 1} \phi(t')\,dt'\,. \tag{20}$$

From now on, for the sake of simplicity, we will assume $q = 1$. The numerical method is implemented in the Fortran routine `voltra.for`, available from

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Numerical Recipes [37]. We note that an alternative semi-analytical method oriented to viscoelasticity has been first introduced by Hopkins and Hamming in the late 1950s [22, 23], and recently revisited by Lin in his book [28]. We also note that, in the limit $\alpha \to -\infty$, $\phi(t) = 1$ and the elastic Hooke model is recovered.

In Figure 2, we show the relaxation function $\phi(t)$ versus the dimensionless time t/τ_0, for some values of α in the interval $-2 \leq \alpha \leq 1$, adopting a logarithmic and a linear time axis in the top and in the bottom frames, respectively.

We conclude this section with a discussion of the asymptotic representations of $\psi(t)$ and $\phi(t)$ as $t/\tau_0 \to 0^+$ and $t/\tau_0 \to +\infty$, which provide simple analytical estimates of the corresponding functions for sufficiently small and large times, respectively. These asymptotic results, indeed trivial for $\psi(t)$, are more relevant for $\phi(t)$, for which we only dispose of a numerical evaluation. From the asymptotic analysis we exclude the trivial case $\alpha = 1$, for which we easily recover the creep and relaxation laws of the Maxwell body, valid for any time:

$$\psi(t) = t/\tau_0, \quad \phi(t) = \exp(-t/\tau_0) \quad t \geq 0. \tag{21}$$

For the function $\psi(t)$, the case $t/\tau_0 \to 0^+$ can be easily studied recalling the binomial series representation

$$\psi(t) = \frac{1}{\alpha} \sum_{n=1}^{\infty} \binom{\alpha}{n} \left(\frac{t}{\tau_0}\right)^n, \tag{22}$$

which is convergent for $0 \leq t/\tau_0 < 1$. Hence, we obtain the straightforward asymptotic result

$$t/\tau_0 \to 0^+ \implies \psi(t) \sim t/\tau_0, \quad \alpha < 1. \tag{23}$$

In the case $t/\tau_0 \to +\infty$, starting from the definition given by Eq. (16), we have:

$$t/\tau_0 \to +\infty \implies \psi(t) \sim \begin{cases} (t/\tau_0)^\alpha/\alpha, & 0 < \alpha < 1, \\ \log(t/\tau_0), & \alpha = 0, \\ \left(1 - (t/\tau_0)^{-|\alpha|}\right)/|\alpha|, & \alpha < 0. \end{cases} \tag{24}$$
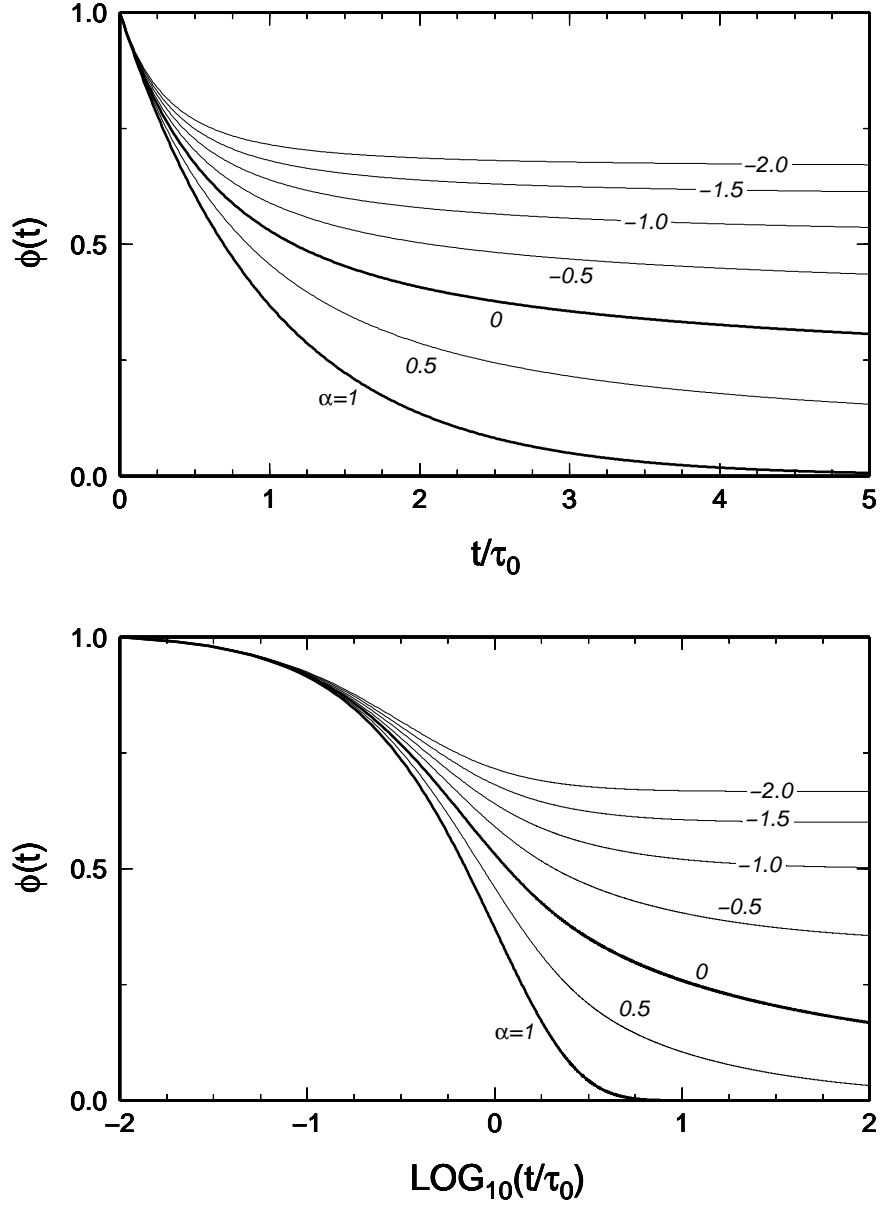
Figure 2: The relaxation function $\phi(t)$ for $\alpha = 1, 0.5, 0, -0.5, -1, -1.5, -2$ versus dimensionless time t/τ_0, adopting a linear scale (top) and a logarithmic scale (bottom).

For the function $\phi(t)$, the asymptotic results for $t/\tau_0 \to 0^+$ and $t/\tau_0 \to +\infty$ can be obtained by applying the Tauberian theorems in the R.H.S. part of Eq. (18) (with $q = 1$) of Laplace transform. Because we prefer to avoid the exact expression of $\widetilde{\psi}(s)$ in terms of transcendental functions, we will use the Tauberian theorems in a simplified form. This means to invert for $s \to \infty$ the Laplace transform of $\psi(t)$ as $t \to 0$, and conversely for $s \to 0$ the Laplace transform of $\psi(t)$ as $t \to \infty$.

As far the behaviour as $t/\tau_0 \to 0^+$ is concerned, from Eqs. (23) and (18) we have:

$$ s \to \infty \implies \widetilde{\psi}(s) \sim 1/s^2 \implies \widetilde{\phi}(s) \sim \frac{1/s}{1 + 1/s} \sim 1/s - 1/s^2 \,, $$

so that

$$ t/\tau_0 \to 0^+ \implies \phi(t) \sim 1 - t/\tau_0 \,, \quad \alpha < 1 \,. \tag{25} $$

As far the behaviour as $t/\tau_0 \to +\infty$ is concerned, from Eqs. (24) and (18) we obtain:

$$ s \to 0 \implies \widetilde{\psi}(s) \sim \begin{cases} \Gamma(\alpha)/s^{\alpha+1} \,, & 0 < \alpha < 1 \,, \\[2ex] -(C + \log s)/s \,, & \alpha = 0 \,, \\[2ex] 1/(|\alpha|s) + \Gamma(-|\alpha|)/s^{-|\alpha|+1} \,, & \alpha < 0 \,, \end{cases} $$

and

$$ s \to 0 \implies \widetilde{\phi}(s) \sim \begin{cases} \dfrac{1/s}{1 + \Gamma(\alpha)/s^{\alpha}} \sim \dfrac{1}{\Gamma(\alpha)s^{1-\alpha}} \,, & 0 < \alpha < 1 \,, \\[3ex] \dfrac{1/s}{1 - C - \log s} \sim \dfrac{1}{s \log(1/s)} \,, & \alpha = 0 \,, \\[3ex] \dfrac{1/s}{1 + 1/|\alpha| + \Gamma(-|\alpha|)/s^{-|\alpha|}} \,, & \alpha < 0 \,, \end{cases} $$

where for the case $\alpha = 0$ the constant $C = -\Gamma'(1) = 0.577215...$ denotes the so called Euler-Mascheroni constant.

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Henceforth

$$
t/\tau_0 \to +\infty \implies \phi(t) \sim
\begin{cases}
\dfrac{\sin(\pi\alpha)}{\alpha}\,(t/\tau_0)^{-\alpha}, & 0 < \alpha < 1, \\[2ex]
\dfrac{1}{\log(t/\tau_0)}, & \alpha = 0, \\[2ex]
\dfrac{1}{1+1/|\alpha|}\left(1+\dfrac{1}{1+|\alpha|}\,(t/\tau_0)^{-|\alpha|}\right), & \alpha < 0,
\end{cases}
\tag{26}
$$

where the Laplace inversion of $1/(s\log(1/s))$ was obtained via the Karamata-Tauberian theory of slow varying functions in the paper by Mainardi et al. [32]. For details on slow varying functions and on Karamata-Tauberian theorems, see *e.g.,* Feller (1971) [17].

4 Retardation spectrum for the extended Jeffreys-Lomnitz law of creep

Following the method outlined in Section 2, see the top Eqs. in (14) or (15), we must consider the derivative of the dimensionless creep function in the extended Jefferys-Lomnitz law as the Laplace transform (with Laplace parameter $\xi = t/\tau_0$) of an "original function" in the variable $\gamma = \tau_0/t$. For sake of simplicity, from now on, we will scale times and frequencies by assuming $\tau_0 = 1$, and also put the material constants J_0 and q equal to 1.

In order to obtain the required time retardation spectrum we first derive the corresponding frequency spectrum for $\alpha < 1$ by proving the Laplace transform pair:

$$
\gamma\, S_\epsilon(\gamma) = \frac{1}{\Gamma(1-\alpha)}\,\frac{\mathrm{e}^{-\gamma}}{\gamma^\alpha} = \div (\xi+1)^{\alpha-1} = \frac{d\psi}{d\xi} = \dot{J}_\tau(\xi), \quad \alpha < 1.
\tag{27}
$$

The prove is based on the well-known integral representation of the Gamma function,

$$
\Gamma(z) := \int_0^\infty \mathrm{e}^{-u}\,u^{(z-1)}\,du, \quad \mathrm{Re}\{z\} > 0.
$$

In fact

$$\frac{1}{\Gamma(1-\alpha)} \int_0^\infty e^{-\xi\gamma} \frac{e^{-\gamma}}{\gamma^\alpha} = \frac{(\xi+1)^{\alpha-1}}{\Gamma(1-\alpha)} \int_0^\infty e^{-u} u^{-\alpha} \, du = (\xi+1)^{\alpha-1}$$

where we have used the change of variable $u = (\xi + 1)\gamma$ and the integral representation of $\Gamma(1 - \alpha)$. Then, from the first expression in Eq. (13) we have

$$R_\epsilon(1/\gamma) = \gamma^2 \, S_\epsilon(\gamma) = \frac{1}{\Gamma(1-\alpha)} \frac{e^{-\gamma}}{\gamma^{\alpha-1}}, \quad \alpha < 1 \,,$$

so that, noting $\tau = 1/\gamma$, we derive the required analytical form for the retardation spectrum,

$$R_\epsilon(\tau) = \frac{1}{\Gamma(1-\alpha)} \frac{e^{-1/\tau}}{\tau^{1-\alpha}}, \quad \alpha < 1 \,. \tag{28}$$

For the limiting values of $\alpha = 1$ (Maxwell body) and $\alpha \to -\infty$ (Hooke body) we recover $R_\epsilon(\tau) = 0$ due to the vanishing of the factor $1/\Gamma(1-\alpha)$.

Accounting the previous notes on the asymptotic behavior of $\psi(t)$ we now explore the integrability of such spectrum on the interval $0 < \tau < \infty$. Using directly Eq. (28), we have

$$\int_0^\infty R_\epsilon(\tau) \, d\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{e^{-u}}{u^{\alpha+1}} \, du = \begin{cases} \infty, & \text{if } 0 \le \alpha < 1 \,, \\ 1/|\alpha| & \text{if } -\infty < \alpha < 0 \,. \end{cases} \tag{29}$$

This is obtained by changing the variable τ into $u = 1/\tau$, recalling the integral representation of the Gamma function and noting $\Gamma(-\alpha) = -\Gamma(1-\alpha)/\alpha$.

Of course the above result is consistent with the asymptotic behaviour of our creep law (16) as $t/\tau_0 \to +\infty$.

In Figure 3, we show the retardation spectrum $R_\epsilon(\tau)$ versus the dimensionless retardation time τ/τ_0 for some values of $\alpha \in [-2, +1]$, adopting in abscissa a logarithmic scale.

We note that, for $\alpha = 0$ (Lomnitz body), the maximum of the spectrum occurs just for $\tau/\tau_0 = 1$. Furthermore, for $\alpha < 0$, as $\tau \to 0$ the maximum increases with $|\alpha|$ up to infinity in such a way that R_ϵ identically vanishes in the open interval $\tau > 0$.
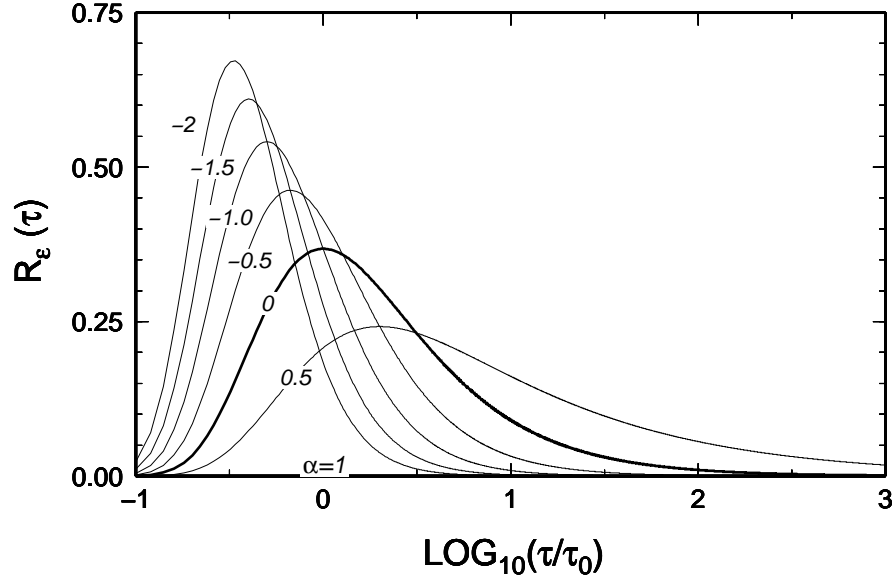
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Figure 3: The retardation spectrum $R_\epsilon(\tau)$ versus dimensionless time τ/τ_0, for $\alpha = 1, 0.5, 0, -0.5, -1, -1.5, -2$, adopting a logarithmic scale.

5   Concluding remarks

By revisiting the Jeffreys-Lomnitz law of creep, we have described the behaviour of a general viscoelastic body that, according to the value of a single parameter α ranging from 1 to $-\infty$, shows a transition from a Maxwell to a Hooke body. This continuous transition is well shown in Figure 1, where the creep law $\psi(t)$ is displayed for some values of α in the range $-2 \leq \alpha \leq 1$ for more rheological interest. We have numerically computed the relaxation modulus $\phi(t)$ corresponding to our creep law by solving a Volterra integral equation of the second kind, see Figure 2. Furthermore, we have provided the analytical expression of the spectrum of retardation times, which is shown in Figure 3.

We are not aware of any previous attempt in the literature to compute the retardation spectrum and the relaxation law for the Jeffreys-Lomnitz law of creep, so we think to have filled a gap in this field. However, we have

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not numerically evaluated the relaxation spectrum leaving this problem to interested readers who can chose the most suitable method to approximate the relaxation spectrum from the numerical data of the relaxation modulus. On this respect there exists a huge literature, see e.g. [1, 2, 3, 4, 11, 12, 13, 14, 16, 18, 28, 35, 39, 45, 46, 47, 48, 51].

We also note that in the recent book by Schilling et al. [40], where a rigorous mathematical description of Bernstein functions is found, the authors have considered a function similar to our extended law of creep in two examples (see No. 2,3 at pp. 218–219, Chapter 15) without the factor $1/\alpha$, corresponding to our parameter $0 < \alpha < 1$ and $-1 < \alpha < 0$. For these cases they have provided the so-called *Lévy density* that, as a matter of fact, coincides with our frequency spectral function $S_\epsilon(\gamma)$ and therefore with our retardation spectrum $R_\epsilon(\tau)$ by putting $\gamma = 1/\tau$. Therefore, in this work, we have extended their results to the cases $\alpha = 0$ and $\alpha \leq -1$ by adopting our notation.

Finally, we hope that the results obtained for the creep and relaxation laws investigated in this paper may be useful for fitting experimental data of creep and/or relaxation responses in rheology of real materials.

Acknowledgments

Appendix: The Post-Widder formula for the retardation spectrum

The Post-Widder formula provides the original function $f(t)$ from its Laplace transform $\widetilde{f}(s)$, known with all its derivatives on the real semi-axis of the

complex Laplace plane, via a limit of infinite sequence as

$$f(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left[s^{n+1} \widetilde{f}^{(n)}(s) \right]_{s=n/t} \qquad (A.1)$$

where $\widetilde{f}^{(n)}$ is the n–th derivative of $\widetilde{f}$ with respect to the Laplace variable s. This is the case when the Laplace transform is proved to be an analytic function on the right-half s-plane. For other details we refer e.g. to [46].

The purpose of this Appendix is to prove the validity of Eq. (27) by using (in a suitable way) the Post-Widder formula. In our case, denoting in Eq. (A.1) t by γ and s by ξ, we must verify that

$$f(\gamma) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left[\xi^{n+1} \widetilde{f}^{(n)}(\xi) \right]_{\xi=n/\gamma} = \frac{1}{\Gamma(1-\alpha)} \frac{e^{-\gamma}}{\gamma^\alpha}, \qquad (A.2)$$

with $\widetilde{f}(\xi) := (\xi + 1)^{\alpha-1}$ and $\alpha < 1$. Now

$$\widetilde{f}^{(n)}(\xi) = (-1)^n \frac{\Gamma(n - \alpha + 1)}{\Gamma(1 - \alpha)} (\xi + 1)^{\alpha-n-1}, \quad n = 1, 2, \ldots \qquad (A.3)$$

so we get

$$\lim_{n \to \infty} \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(n - \alpha + 1)}{\Gamma(n + 1)} \left(\frac{n}{\gamma} \right)^{n+1} \left(1 + \frac{n}{\gamma} \right)^{\alpha-n-1}. \qquad (A.4)$$

We easily recognize from Stirling asymptotic formula that for $n \to \infty$

$$\frac{\Gamma(n - \alpha + 1)}{\Gamma(n + 1)} \sim n^{-\alpha}. \qquad (A.5)$$

Because

$$\left(\frac{n}{\gamma} \right)^{n+1} \left(1 + \frac{n}{\gamma} \right)^{\alpha-n-1} = \left(\frac{n}{\gamma} \right)^{\alpha} \left(1 + \frac{\gamma}{n} \right)^{\alpha-n-1} \qquad (A.6)$$

we finally get

$$f(\gamma) = \frac{1}{\Gamma(1-\alpha)\gamma^\alpha} \lim_{n \to \infty} \left(1 + \frac{\gamma}{n} \right)^{-n} \left(1 + \frac{\gamma}{n} \right)^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)} \frac{e^{-\gamma}}{\gamma^\alpha}, \quad (A.7)$$

in view of the Neper limit

$$\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e. \qquad (A.8)$$

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References

[1] Z.P. Bažant and S-S. Kim, Approximate relaxation function for concrete, *J. Struct.* ASCE **105**, 2695–2705 (1979).

[2] Z.P. Bažant and S. Prasannan, Solidification theory for concrete creep. I: Formulaton, *Int. Engrg. Mech.* ASCE **115** No 8, 1691–1703 (1989).

[3] Z.P. Bažant and S. Prasannan, Solidification theory for concrete creep. II: Verification and application, *Int. Engrg. Mech.* ASCE **115** No 8, 1704–1725 (1989).

[4] Z.P. Bažant and Yu. Xi, Continuous retardation spectrum for solidification theory of concrete creep, *I. Engrg. Mech.* ASCE **121**, 281–288 (1995).

[5] Ch. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups* (Springer, Berlin, 1975).

[6] D. R. Bland, *The Theory of Linear Viscoelasticity* (Pergamon, Oxford, 1960).

[7] R. M. Christensen, *Theory of Viscoelasticity*, Second edition, (Academic Press, New York, 1982).

[8] S.T. Crough and R.O. Burford, Empirical law for fault-creep events, *Tectonophisics* **42**, T53–T59 (1977).

[9] D. J. Darby and E. G. C. Smith, Power-law stress relaxation after the 1987 Edgecumbe, New Zealand, earthquake, *Geophys. J. Intl.* **103**, 561–563 (1990).

[10] G. Del Piero and L. Deseri, Monotonic, completely monotonic and exponential relaxation in linear viscoelasticity, *Quart. Appl. Math.* **53**, 273–300 (1995).

[11] I. Emri, *Rheology of Solid Polymers*, (Rheology Reviews, British Society of Rheology, 2005).

[12] I. Emri and N.W. Tschoegl, Generating line spectra from experimental responses, Part I: Relaxation modulus and creep compliance. *Rheol. Acta* **32**, 311–321 (1993).

[13] I. Emri and N.W. Tschoegl, Generating line spectra from experimental responses, Part IV: Application to experimental data, *Rheol. Acta* **33**, 60–70 (1994).

[14] I. Emri and N.W. Tschoegl, Generating line spectra from experimental responses, Part V: Time-dependent viscosity *Rheol. Acta* **36**, 303–306 (1997).

[15] I. Emri and N.W. Tschoegl, Determination of mechanical spectra from experimental responses and of the steady-state compliance, *Int. J. Solids Structures* **32** No 6/7, 817–826 (1995).

[16] I. Emri, B.S von Bernstorff, R. Cvelbar and A. Nikonov, Re-examination of the approximation methods for interconversion between frequency- and time-dependent material functions, *J. Non-Newtonian Fluid Mech.* **129**, 75–84 (2005).

[17] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II, Second Edition (Wiley, New York, 1971).

[18] J.D. Ferry, *Viscoelastic Properties of Polymers*, Second edition (Wiley, New York, 1980).

[19] G. Gripenberg, S.-O. Londen and O.J. Staffans, *Volterra Integral and Functional Equations* (Cambridge University Press, Cambridge, 1990).

[20] B. Gross, *Mathematical Structure of the Theories of Viscoelasticity* (Hermann, Paris, 1953).

[21] A. Hanyga, Viscous dissipation and completely monotonic relaxation moduli, *Rheologica Acta* **44**, 614–621 (2005).

[22] I.L. Hopkins and R.W. Hamming, On Creep and Relaxation, *J. Appl. Phys.* **28**, 906–909 (1957).

[23] I.L. Hopkins and R.W. Hamming, Note on paper "On Creep and Relaxation", *J. Appl. Phys.* **29**, 742 (1958).

[24] H. Jeffreys, A modification of Lomnitz's law of creep in rocks, *Geophys. J. R. Astron. Soc.* **1**, 92–95 (1958).

[25] H. Jeffreys *The Earth*, 6-th Ed. (Cambridge University Press, Cambridge, 1976).

[26] H. Jeffreys and S. Crampin, Rock creep: a correction, *Mon. Not. R. Astron. Soc.* **121**, 571–577 (1960).

[27] H. Jeffreys and S. Crampin, On the modified Lomnitz law of damping, *Mon. Not. R. Astron. Soc.* **147**, 295–301 (1970).

[28] Yn-H. Lin, *Polymer Viscoelasticity*, 2-nd Ed. (World Scientific, Singapore, 2011).

[29] C. Lomnitz, Creep measurements in igneous rocks, *J. Geol.* **64**, 473–479 (1956).

[30] C. Lomnitz, Linear dissipation in solids, *J. Appl. Phys.* **28**, 201–205 (1957).

[31] C. Lomnitz, Application of the logarithmic creep law to stress wave attenuation in the solid Earth, *J. Geophys. Res.* **67**, 365–367 (1962).

[32] F. Mainardi, A. Mura, R. Gorenflo and M. Stojanovic, The two forms of fractional relaxation of distributed order, *Journal of Vibration and Control* **13**, 1249–1268 (2007). [E-print http://arxiv.org/abs/cond-mat/0701131]

[33] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, (Imperial College Press–World Scientific, London, 2010).

[34] A. Molinari, Viscoélasticité linéaire et function complètement monotones, *Journal de Mécanique* **12**, 541–553 (1975).

[35] S.W. Park and R.A. Schapery, Methods of interconversion between linear viscoelastic material functions. Part I–a numerical method based on Prony series, *International Journal of Solids and Structures* **36**, 1653-1675 (1998).

[36] A. C. Pipkin, *Lectures on Viscoelastic Theory*, 2-nd Ed. (Springer, New York, 1986).

[37] W.H. Press, S.A. Teukolsky, W.T. Vetterling and B.P. Flannery, *Fortran Recipes in Fortran 77 - The Art of Scientific Computing* (Cambridge University Press, Cambridge, 1986).

[38] G. Ranalli, *Rheology of the Earth* (Allen & Unwin, London, 1987).

[39] R.A. Schapery and S.W. Park, Methods of interconversion between linear viscoelastic material functions. Part II- an approximate analytical method, *International Journal of Solids and Structures* **36**, 1677-1699 (1999).

[40] R. L. Schilling, R. Song and Z. Vondracek, *Bernstein Functions: Theory and Applications* (De Gruyter, Berlin, 2010).

[41] J.W. Spencer, Stress relaxation at low frequencies in fluid-saturated rocks: attenuation and modulus dispersion, *J. Geophys. Res.* **86**, 1803-1812 (1981).

[42] E. Strick, Applications of linear viscoelasticity to seismic wave propagation, in F. Mainardi (Editor), *Wave Propgation in Viscoelstic Media*, Res. Notes in Math. 52, pp. 169–182 (Pittman, London, 1982).

[43] E. Strick, Implications of Jeffreys–Lomnitz transient creep, *J. Geophys. Res.* **89**, 437–451 (1984).

[44] E. Strick and F. Mainardi, On a general class of constant-Q solids, *Geophys. J. R. Astron. Soc.* **69**, 415–429 (1982).

[45] N.W. Tschoegl, A general method for the determination of approximations to the spectral distributions from transient response functions, *Rheol. Acta* **10** 595–600 (1971).

[46] N.W. Tschoegl, *The Phenomenological Theory of Linear Viscoelastic Behavior: an Introduction* (Springer Verlag, Berlin, 1989).

[47] N.W. Tschoegl, Time dependence in material properties: an overview, *Mechanics of Time-Depenent Materials* **1** 3–31 (1997).

[48] N.W. Tschoegl and I. Emri, Generating line spectra from experimental responses, Part III: Interconversion between relaxation and retardation behavior, *International Journal of Polymeric Materials* **18** No 1-2, 117–127 (1992).

[49] P. Wessel and W.H.F. Smith, New, improved version of the Generic Mapping Tools released, *Eos Trans. Amer. Geophys. Union* **79**, 579 (1998).

[50] R.L. Wesson, Dynamics of fault creep, *J. Geophys. Res.* **93**, 8929–8951 (1988).

[51] G. Zi and Z.P. Bažant, Continuous relaxation spectrum of concrete creep and its incorporation into microplane model M4, in F.-J. Ulm, Z.P. Bažant and F.H. Wittmann, *Creep, Shrinkage and Durability Mechanics of Concrete and other Quasi-Brittle Materials* (Elsevier, Amsterdam, 2001), pp. 239–243.